

Entanglement dynamics in a non-Markovian environment: an exactly solvable model

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We study the non-Markovian effects on the dynamics of entanglement in an exactly-solvable model that involves two independent oscillators each coupled to its own stochastic noise source. First, using Lie algebraic and functional integral methods, we present an exact solution to the single-oscillator problem which includes an analytic expression for the density matrix and the complete statistics, i.e., the probability distribution functions for observables. We see non-monotonic evolution of the uncertainties in observables. Further, we extend this exact solution to the two-particle problem and calculate the entanglement in a subspace. We find the phenomena of ‘sudden death’ and ‘rebirth’ of entanglement. Interestingly, the time of death and rebirth is controlled by the amount of ‘noisy’ energy added into each single oscillator. If this energy increases above (decreases below) a threshold, we obtain sudden death (rebirth) of entanglement.

Noise in quantum systems can lead to abrupt and complete destruction (sudden death) of entanglement [1]. This is troubling since the death (sudden or otherwise) of entanglement represents one of the major obstacles towards building a quantum computer; see for example [2]. In particular, when the bath is Markovian (memoryless as in [1]), the death of entanglement can be rather swift since the memory of the system’s quantum state is wiped away by its totally uncorrelated interactions with the bath.

Entanglement dynamics including sudden death and birth has been studied theoretically, e.g., in two qubits in several contexts [1, 3–5] and in harmonic oscillators [6]. The recent observation of these phenomena in photonic systems [7] and ensembles of atoms [8] has attracted great interest. In particular, it has been suspected that bath memory effects could not only provide an avenue to prolong entanglement but could also lead to its rebirth after it has experienced sudden death [3]. However, most noisy environments are hard to treat analytically by standard techniques [9] and one must use numerics or impose approximations to obtain a tractable result.

In this Letter, we present an exactly solvable model involving two independent harmonic oscillators each interacting with its own classical non-Markovian stochastic reservoir. No back-reaction to the reservoirs is considered. This system has the property that it can be solved analytically allowing us to study non-Markovian effects on the dynamics of entanglement including the prolonging of entanglement and its rebirth. Particularly, we study the dynamics of entanglement for the lowest two states of the oscillators which form a qubit-like system. Curiously, there is a one-to-one correspondence between the amount of “noisy” energy added to each oscillator and their entanglement: As the energy increases (decreases) across a threshold, we see sudden death (rebirth) of entanglement (see Fig. 1). Furthermore, this initial-state dependent threshold is *independent* of the form of the noise correlations in time.

The main advantage of our model is that it is exactly

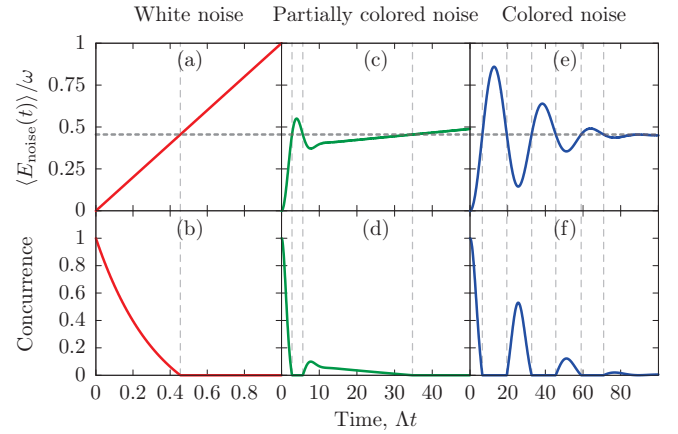


FIG. 1. (Color online) Comparison between noisy energy of one of the oscillators and concurrence (entanglement) for baths with different memory. The initial state is $(|01\rangle + |10\rangle)/\sqrt{2}$. When the energy exceeds (falls below) the threshold 0.455ω , we see sudden death (rebirth) of entanglement. $\langle E_{\text{noise}}(t) \rangle = \langle \hat{E}(t) \rangle_{\xi} - \langle \hat{E}(0) \rangle_{\xi}$ where $\langle \hat{E}(t) \rangle_{\xi}$ is the expectation value of the energy of a single oscillator averaged over noise. (a,b) use $\Lambda\tau = 0$; (c,d) $\Lambda\tau = 4$ and $\omega/\Lambda = 0.875$; (e,f) $\Lambda\tau = 30$ and $\omega/\Lambda = 0.25$.

solvable and non-Markovian [10]. We provide exact analytical expressions for a single oscillator’s probability distribution function (PDF) of position, momentum, and energy where the memory of the bath is taken into account explicitly. We find that the width of the PDFs of position, momentum, and energy are oscillatory in non-Markovian environments. Interestingly, this means that non-Markovian environments do not necessarily add uncertainty to the system monotonically. We also provide exact expressions for the correlation functions of observables. In this situation we characterize the crossover from Markovian to non-Markovian behavior.

Entanglement in harmonic oscillators can be quantified in several ways [6] and can be produced on demand with trapped ion systems [11]. Here, we focus on the low-

est two states of each oscillator which form a two-qubit Hilbert subspace. For a two-qubit system, entanglement is unambiguously quantified in terms of the *concurrence* $C(\hat{\rho}_2(t))$, where $\hat{\rho}_2$ is the density matrix of two two-level system, we have

$$C(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (1)$$

where λ_i are the eigenvalues (in decreasing order) of the matrix $\hat{\rho}_2(t)\tilde{\rho}_2(t)$ where $\tilde{\rho}_2 = (\sigma_y \otimes \sigma_y)\hat{\rho}_2^*(\sigma_y \otimes \sigma_y)$. Physically, it can be shown [12] that states are maximally entangled if $C(t) = 1$ and completely unentangled for $C(t) = 0$. When $C(t) = 0$ there exists a realization of $\hat{\rho}_2(t)$ such that $\hat{\rho}_2(t) = \sum_k p_k |\psi_k\rangle \langle \psi_k|$ where every $|\psi_k\rangle$ is separable; i.e., the system is a classical mixture of separable states. The concurrence can vanish or appear suddenly at a finite time, counter to what one may naively expect from the exponential decay of coherences (with characteristic time T_2) which are local quantum phenomena.

Consider the Hamiltonian of two harmonic oscillators $\hat{H} = \hat{H}_1 + \hat{H}_2$ in the presence of two statistically independent sources of classical noise ($m = 1, 2$)

$$\hat{H}_m = \omega(a_m^\dagger a_m + \frac{1}{2}) + \frac{1}{\sqrt{2}}[\xi^{(m)}(t)a_m^\dagger + \text{h.c.}], \quad (2)$$

(no sum over m) where $a_m^\dagger(a_m)$ are the standard creation (annihilation) operators with $[a_n, a_m^\dagger] = \delta_{nm}$ and $\xi^{(m)}(t) = \xi_1^{(m)}(t) + i\xi_2^{(m)}(t)$ are external stochastic fields with correlation functions of the form

$$\begin{aligned} \langle \xi_i^{(n)}(t)\xi_j^{(m)}(t') \rangle_\xi &= \delta_{nm}K_{ij}(t, t') = \delta_{nm}\delta_{ij}k(t - t') \\ &= \delta_{nm}\delta_{ij}\frac{\Lambda}{\tau\sqrt{2\pi}}e^{-(t-t')^2/2\tau^2}, \end{aligned} \quad (3)$$

where the average over noise is defined as the functional integral, $\langle(\dots)\rangle_\xi = \int \mathcal{D}^2\xi(\dots)\exp[-\frac{1}{2}\int\int\xi^TK^{-1}\xi]$ [10] (suppressing the matrix indices for clarity). Our conclusions do not depend on the explicit form for the correlation function $k(t - t')$ but on the time scale of decay of correlations τ and the amplitude of the noise Λ . For $\tau = 0$, the bath has no memory and this leads to well known Markovian behavior [9]. We are mostly concerned with the regime where $\tau \neq 0$.

We begin our analysis considering non-Markovian effects on a *single* oscillator, i.e., H_1 from Eq. (2). We calculate the PDF of position, momentum, and energy with Gaussian noise [10]. For an operator \hat{A} its PDF $P_{\hat{A}}(A; t)$ is the noise averaged quantity $P_{\hat{A}}[A; t] = \langle \delta[A - \langle \hat{A}(t) \rangle] \rangle_\xi$. We find the expectation values of position and momentum ($\langle \hat{x}(t) \rangle$ and $\langle \hat{p}(t) \rangle$ respectively) to be normally distributed about the solutions of the classical equations of motion: $X_{\text{cl}}(t) = x_0 \cos \omega t + p_0 \sin \omega t$ and $P_{\text{cl}}(t) = -x_0 \sin \omega t + p_0 \cos \omega t$. Given an initial coherent state of the form $|\psi(0)\rangle = |z_0\rangle$ where $z_0 = (x_0 + ip_0)/\sqrt{2}$, explicit

calculation gives

$$P_{\hat{x}}[X; t] = \frac{1}{\sqrt{2\pi}\Sigma_x(t)} \exp\left\{-\frac{[X - X_{\text{cl}}(t)]^2}{2\Sigma_x(t)^2}\right\}, \quad (4)$$

$$P_{\hat{p}}[P; t] = \frac{1}{\sqrt{2\pi}\Sigma_p(t)} \exp\left\{-\frac{[P - P_{\text{cl}}(t)]^2}{2\Sigma_p(t)^2}\right\}, \quad (5)$$

where X, P are random variables and the width of the PDFs is

$$\Sigma_{x,p}^2(t) = \int_0^t \int_0^t ds ds' g_i^{x,p}(t, s) K_{ij}(s, s') g_j^{x,p}(t, s'), \quad (6)$$

using Einstein summation convention from here on. We defined $g_1^x(t, s) = -\sin \omega(t - s)$, $g_1^p(t, s) = -\cos \omega(t - s)$, $g_2^x(t, s) = \cos \omega(t - s)$, $g_2^p(t, s) = -\sin \omega(t - s)$. Notice that these Gaussian PDFs for both the position and momentum are centered about the classical path and spread out in a Gaussian manner from that. From this, we can see that $\langle \langle \hat{x}(t) \rangle \rangle_\xi$ and $\langle \langle \hat{p}(t) \rangle \rangle_\xi$ will satisfy the standard classical equations of motion for the harmonic oscillator. Additionally, from Eq. (6) we can study the effects of the memory of the bath. For delta-correlated noise ($\tau = 0$) we find Brownian growth as $\Sigma_{x,p}(t) = \sqrt{\Lambda t}$ which is expected for Markovian-type of behavior. If we include memory in the bath we obtain

$$\Sigma_{x,p}^2(t) = \epsilon(t) = \int_0^t \int_0^t ds ds' \cos \omega(s - s') k(s - s'), \quad (7)$$

which defines $\epsilon(t)$, Fig. 2; this function appears frequently in our analysis. A change of variables gives

$$\epsilon(t) = t \int_{-t}^t du k(u) \cos \omega u - \int_{-t}^t du |u| k(u) \cos \omega u. \quad (8)$$

The first term in Eq. (8) gives the slope of $\epsilon(t)$ (and hence $\Sigma_{x,p}^2(t)$) as $t \rightarrow \infty$. For Gaussian time correlations $k(t) = (\Lambda/\tau\sqrt{2\pi})\exp[-t^2/2\tau]$ we see that the slope is $\Lambda \exp[-\omega^2\tau^2/2]$ as $t \rightarrow \infty$ and the second derivative, at any time, is $(2\Lambda/\tau\sqrt{2\pi})\cos \omega t \exp[-t^2/2\tau^2]$, so for short times the behavior is $\epsilon(t) \rightarrow (2\Lambda/\tau\sqrt{2\pi})t^2$. Notice, the slope at long times vanishes exponentially as τ increases.

The non-monotonic behavior in $\Sigma_{x,p}^2(t)$ means that the amount of spreading in the expectation value of \hat{x} and \hat{p} can decrease at short times if the bath is non-Markovian ($\tau \neq 0$) counter to what one would naively expect given a noisy environment. At large times it grows linearly in time. This spreading is distinct, and in addition to the natural quantum mechanical spread of the system observables which is not captured by the PDFs. The non-Markovian effects can actually decrease the randomness in the system for short periods of time. In the extreme case of non-decaying noise correlations: $K_{ij} = \Lambda\delta_{ij}$, $\Sigma_{x,p}^2(t) = 2\Lambda(1 - \cos \omega t)/\omega^2$ which means that the PDFs $P_{\hat{x}}[X, 2\pi k/\omega]$ and $P_{\hat{p}}[P, 2\pi k/\omega]$ (k is an integer) collapse into delta functions. At these discrete

times, the expectation value of these observables will yield the classical value of position and momentum with probability one and purely quantum mechanical behavior is restored. Intuitively, the system remembers its initial pure state and tries to restore it. When the memory is not long, the restoration is not complete but still can give non-monotonic behavior.

We can also evaluate the PDF for the number operator analytically given an initial state Fock state $|n\rangle$. Explicit calculation [10] gives an expression that depends only on $\epsilon(t)$,

$$P_{\hat{n}}[N; t] = \frac{\Theta(N - n)}{\epsilon(t)} e^{-(N-n)/\epsilon(t)}, \quad (9)$$

where N is the (continuous since it is expectation value of \hat{n}) number random variable and Θ is the step function. Eq. (9) implies that only states with energy above the initial state $|n\rangle$ are occupied due to the heating. The exponential PDF has mean $n + \epsilon(t)$ and variance $\epsilon(t)^2$ with the memory of the bath being taken into account via $\epsilon(t)$; see Fig. 2. The non-Markovian effects will cause the PDF to narrow as well, and in the limit of infinite time correlations of the bath it will periodically return to $\delta(N - n)$.

All single oscillator statistical properties can be obtained from our PDFs. Non-Markovian effects also arise in the correlation functions of position and momentum. For example, if we let $\hat{X}(t) = [\hat{x}(t), -\hat{p}(t)]^T$ and assuming that the initial state is $|n\rangle$ we can write $\hat{X}_i(t) = \hat{X}_{0i}(t) + \int_0^t ds R_{ij}(t, s) \xi_j(s)$ where $\hat{X}_0(t)$ is the evolution without any driving force: $\hat{x}_0(t) = (e^{-i\omega t} a + e^{i\omega t} a^\dagger)/\sqrt{2}$ and $\hat{p}_0(t) = (e^{-i\omega t} a - e^{i\omega t} a^\dagger)/i\sqrt{2}$. With these definitions the noise-averaged correlation functions are

$$\begin{aligned} \langle \langle \hat{X}_i(t) \hat{X}_j(t') \rangle \rangle_\xi &= \langle \hat{X}_{0i}(t) \hat{X}_{0j}(t') \rangle + \int_0^t ds \int_0^{t'} ds' \\ &\times R_{ik}(t, s) K_{kl}(s, s') R_{lj}(s', t'), \end{aligned} \quad (10)$$

where $R_{11}(t, t') = R_{22}(t, t') = \cos \omega(t - t')$, $R_{12}(t, t') = -R_{21}(t, t') = \sin \omega(t - t')$. $\langle \dots \rangle$ is the quantum mechanical expectation value and $\langle \dots \rangle_\xi$ is the average over noise. The average of the energy, $\langle \langle \hat{E}(t) \rangle \rangle_\xi / \omega = (1/2) \langle \langle \hat{x}^2(t) + \hat{p}^2(t) \rangle \rangle_\xi = (1/2) \sum_i \langle \langle \hat{X}_i(t) \hat{X}_i(t) \rangle \rangle_\xi$ is

$$\begin{aligned} \langle \langle \hat{E}(t) \rangle \rangle_\xi / \omega - (n + 1/2) &\equiv \langle E_{\text{noise}}(t) \rangle / \omega \\ &= \frac{1}{2} \int_0^t \int_0^t ds ds' \text{tr} R(t, s) K(s, s') R(s', t). \end{aligned} \quad (11)$$

where we suppress matrix indices for clarity. Eq. (11) defines $\langle E_{\text{noise}}(t) \rangle$. Using Eq. (3) and defining $\Delta A = \hat{A} - \langle \hat{A} \rangle$ we obtain

$$\langle \langle \Delta p(t)^2 \rangle \rangle_\xi = \langle \langle \Delta x(t)^2 \rangle \rangle_\xi = \frac{1}{2} (n + 1/2) + \epsilon(t), \quad (12)$$

$$\langle E_{\text{noise}}(t) \rangle = \omega \epsilon(t), \quad (13)$$

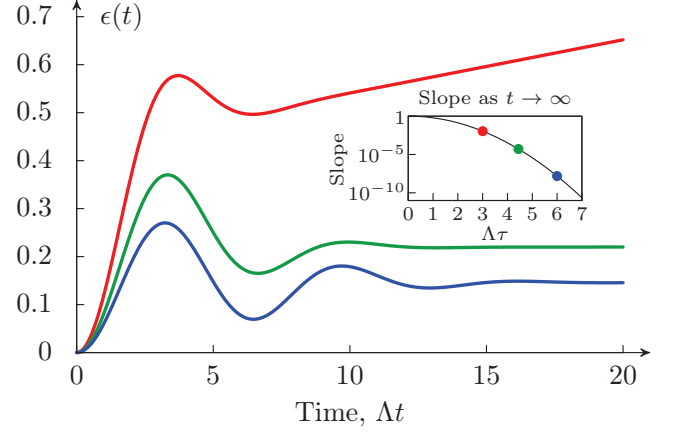


FIG. 2. (Color online) The function $\epsilon(t)$ as defined in Eq. (7) with Gaussian time-correlations. This quantity is a stand-in for: the rise in energy due to the stochastic forcing $\langle E_{\text{noise}}(t) \rangle$ (Eq. (13)), the variances in \hat{x} and \hat{p} averaged over time (Eqs. (12)), and the width of the PDF of position, momentum $\Sigma_{x,p}^2(t)$ (Eq. (7)), and energy (Eq. (9)). From top to bottom: $\omega\tau = 3$, $\omega\tau \approx 4.44$, and $\omega\tau = 6$; in each case we have used $\omega = \Lambda$.

We see linear Brownian-type behavior at large times (for all bath correlation times τ) with the variance of position, momentum and excess (noisy) energy of the system. This is explicit in Fig. 1a and Fig. 2. We also see oscillations at short times for correlated baths [13]. Specifically, Fig. 1c shows an example for characteristic parameters $\omega\tau \approx 3.5$ corresponding to moderate non-Markovian and Fig. 1e with $\omega\tau \approx 7.5$ corresponding to a strongly non-Markovian regime. We see a continuous cross-over from Markovian behavior, Fig. 1a, to strongly non-Markovian behavior, Fig. 1e, where the slope at long times vanishes exponentially, Fig. 2.

Next, we calculate the single oscillator density matrix to capture both quantum and statistical properties of our system. We start by exactly computing the density matrix $\hat{\rho}_1(t)$ for one oscillator averaged over noise realizations. We define the density matrix using the quantum ‘Liouvillian’ $\mathcal{L}(t)$ as

$$\hat{\rho}_1(t) = e^{\mathcal{L}(t)} \hat{\rho}_1(0) \equiv \langle \hat{U}(t) \hat{\rho}_1(0) \hat{U}^\dagger(t) \rangle_\xi, \quad (14)$$

where $\hat{\rho}_1(0) = |\psi(0)\rangle \langle \psi(0)|$ is the initial density matrix and $\hat{U}(t)$ is the single oscillator evolution operator which is the solution of $i\partial_t \hat{U}(t) = \hat{H}_1(t) \hat{U}(t)$, $\hat{U}(0) = 1$ and can be exactly solved [14]. The density matrix can be expanded as $\hat{\rho}_1(0) = \sum_{nm} a_{mn} |m\rangle \langle n|$ and therefore we only need to calculate the evolution of the basis elements

$|m\rangle\langle n|$. Explicit calculation [10] gives

$$e^{\mathcal{L}(t)}|m\rangle\langle n| = e^{-i\omega t(m-n)} \sum_{k,l=0}^{\infty} \delta_{l+m,k+n} \sqrt{\frac{n!l!}{m!k!}} \\ \times \frac{\epsilon(t)^{m+k} {}_2F_1[-k, -m; 1+n-m, \epsilon(t)^{-2}]}{(n-m)!(1+\epsilon(t))^{n+k+1}} |k\rangle\langle l|, \quad (15)$$

for $n \geq m$ and ${}_2F_1$ is the hypergeometric function. For $m \geq n$ exchange $m \leftrightarrow n$ and $k \leftrightarrow l$ on the right hand side of Eq. (15) (except in the $e^{-i\omega t(m-n)}$ phase factor) [10]. Next, we use this expression to compute the density matrix $\hat{\rho}(t)$ for two uncoupled entangled oscillators with Hamiltonian $\hat{H} = \hat{H}_1 + \hat{H}_2$, Eq. (2),

$$\hat{\rho}(t) = e^{\mathcal{L}_1(t)} \otimes e^{\mathcal{L}_2(t)} \hat{\rho}(0), \quad (16)$$

with the initial density matrix $\hat{\rho}(0) = |\psi_0\rangle\langle\psi_0|$ where

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad (17)$$

and the states $|nm\rangle$ represents the first oscillator in state $|n\rangle$ and the second in state $|m\rangle$. The density matrix can be written as $\hat{\rho}(t) = \sum_{nm,n'm'} \langle nm|\hat{\rho}(t)|n'm'\rangle |nm\rangle\langle n'm'|$. We are only interested in how the qubit-like entanglement in the subspace $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ evolves in time. This defines a new 4×4 density matrix $\hat{\rho}_2$ given by $\hat{\rho}_2(t) = \hat{\Pi}\hat{\rho}(t)\hat{\Pi}$ where $\hat{\Pi} = \sum_{n,m=0}^1 |nm\rangle\langle nm|$ is the projection operator onto the subspace. We normalize this expression by the trace of $\hat{\Pi}\hat{\rho}(t)\hat{\Pi}$ for convenience, but this does not effect our conclusions. Explicit calculation [10] yields

$$\hat{\rho}_2 = \frac{1}{(1+\epsilon(t))^3} \\ \times \begin{pmatrix} \epsilon(t) & 0 & 0 & 0 \\ 0 & \frac{\frac{1}{2}+\epsilon(t)^2}{1+\epsilon(t)} & \frac{1/2}{1+\epsilon(t)} & 0 \\ 0 & \frac{1/2}{1+\epsilon(t)} & \frac{\frac{1}{2}+\epsilon(t)^2}{1+\epsilon(t)} & 0 \\ 0 & 0 & 0 & \frac{\epsilon(t)(1+\epsilon(t)^2)}{(1+\epsilon(t))^2} \end{pmatrix}. \quad (18)$$

Given this density matrix we compute the concurrence as given in Eq. (1). The results are presented in Fig. 1 along with plots of the energy added to a *single* oscillator. To see the connection between concurrence and energy, it can be shown directly that the energy given to a single oscillator by the stochastic forcing field is again Eq. (13) (precisely: the energy is the average of the energies for $|0\rangle$ and $|1\rangle$ time evolved separately). Eq. (18) and Eq. (13) both depend on $\epsilon(t)$, hence the energy controls the concurrence. Thus in Fig. 1 we show the behavior of the energy and concurrence on different time scales for the memory of the bath. We see that for a bath with no memory (white noise) the energy increasing linearly as a function of time and the concurrence vanishing at a critical time $At_c \approx 0.455$. For a bath with memory, non-Markovian oscillations of the energy in time lead to a rebirth of the entanglement as the energy crosses a specific

initial-state dependent but noise-independent threshold $\langle E_{\text{noise}} \rangle / \omega \approx 0.455$. Intuitively, the system ‘remembers’ its quantum state, particularly its entanglement. This rebirth phenomenon is absent in baths with no memory (Fig. 1a, 1b).

In terms of our initial density matrix, we may generate entanglement between higher energy states. Letting $\hat{P} = 1 - \hat{\Pi}$ be the projection onto the rest of the Hilbert space, then schematically we have $\hat{\rho} = \hat{\Pi}\hat{\rho}\hat{\Pi} + \hat{P}\hat{\rho}\hat{\Pi} + \hat{\Pi}\hat{\rho}\hat{P} + \hat{P}\hat{\rho}\hat{P}$, and only the first term $\hat{\Pi}\hat{\rho}\hat{\Pi}$ is separable when $C(t) = 0$ (precisely: it can be written as the sum of density matrices of separable states) while the higher energy states may still exhibit entanglement between themselves and the lower energy states. Intuitively, the higher energy states act as a ‘cavity’ to their respective ‘qubit’ (as in [5]), so one may expect entanglement is being transferred back and forth between them (as the classical noise slowly kills the total entanglement).

In summary, we study *analytically* sudden death and rebirth of entanglement in a subspace of two entangled oscillators, and we showed that this is related to the memory of the bath. The longer the memory, the more the system ‘remembers’ its quantum state. Further, this effect is directly controlled by when the amount of ‘noisy’ energy entering the system crosses a initial-state-dependent threshold. The same non-Markovian behavior is also responsible for a narrowing of the probability distribution function of position, momentum, and energy of a single oscillator at short times.

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SUPPLEMENTARY MATERIAL

SYSTEM AND DEFINITIONS

The Hamiltonian for a forced harmonic oscillator is ($\hbar = 1$)

$$\hat{H} = \omega(a^\dagger a + \frac{1}{2}) + \frac{1}{\sqrt{2}}[\xi(t)a^\dagger + \xi^*(t)a]. \quad (1)$$

where $\xi(t) = \xi_1(t) + i\xi_2(t)$ is a stochastic field, and $a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$ where a^\dagger , a are the standard bosonic creation and annihilation operators. The noise correlator that fully characterizes our Gaussian noise is

$$\langle \xi_i(t)\xi_j(t') \rangle_\xi = K_{ij}(t, t') = \delta_{ij}k(t - t'). \quad (2)$$

Averaging over the ξ -fields can be done by path integration

$$\langle (\dots) \rangle_\xi = \frac{\int \mathcal{D}^2\xi (\dots) e^{-\frac{1}{2} \int_0^t dt' \int_0^t dt'' \xi_i(t') K_{ij}^{-1}(t', t'') \xi_j(t'')}}{\int \mathcal{D}^2\xi e^{-\frac{1}{2} \int_0^t dt' \int_0^t dt'' \xi_i(t') K_{ij}^{-1}(t', t'') \xi_j(t'')}} \quad (3)$$

The evolution opertor for a single harmonic oscillator, $i\partial_t \hat{U}(t) = \hat{H}(t)\hat{U}(t)$ with $\hat{U}(0) = \hat{1}$, is [14]

$$\hat{U}(t) = e^{-i\omega t(\hat{n}+1/2)} e^{-i(\Phi_1(t)\hat{x} + \Phi_2(t)\hat{p})} e^{i\gamma(t)}. \quad (4)$$

where $\Phi_i(t) = \int_0^t dt' \xi_j(t') R_{ji}(t')$ (Einstein summation convention used throughout), $\hat{n} = a^\dagger a$, and R is the 2×2 rotation matrix

$$R(t) = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \quad (5)$$

The phase $\gamma(t)$ will be inconsequential for our calculations. It will further be useful define the adjoint of an operator $\text{ad}_{\hat{X}} \hat{Y} = [\hat{X}, \hat{Y}]$.

DENSITY MATRIX

We compute the density matrix of a single oscillator by averaging over noise realizations of $\xi_{1,2}(t)$. Using Eq. (4) and $e^{\hat{X}} \hat{Y} e^{-\hat{X}} = e^{\text{ad}_{\hat{X}}} \hat{Y}$ we obtain

$$\hat{\rho}(t; \xi(t)) = e^{-i\omega t \hat{n}} e^{-i(\Phi_1(t)\hat{x} + \Phi_2(t)\hat{p})} \hat{\rho}_0 e^{i(\Phi_1(t)\hat{x} + \Phi_2(t)\hat{p})} e^{i\omega t \hat{n}} \quad (6)$$

$$= e^{-i\omega t \text{ad}_{\hat{n}}} e^{-i(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} \hat{\rho}_0. \quad (7)$$

where $\hat{\rho}_0$ is the initial density matrix. The noise-averaged density matrix is

$$\hat{\rho}(t) = \langle \hat{\rho}(t; \xi(t)) \rangle_\xi \quad (8)$$

$$= e^{-i\omega t \text{ad}_{\hat{n}}} \langle e^{-i(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} \rangle_\xi \hat{\rho}_0. \quad (9)$$

Let $\hat{x}_1 \equiv \hat{x}$ and $\hat{x}_2 \equiv \hat{p}$. Note further, $[\text{ad}_{\hat{x}}, \text{ad}_{\hat{p}}] = 0$ and $[\text{ad}_{\hat{x}}, 1] = 0 = [\text{ad}_{\hat{p}}, 1]$ allow us to treat $\text{ad}_{\hat{x}}$ and $\text{ad}_{\hat{p}}$ as c -numbers when integrating over $\xi_{1,2}$:

$$\hat{\rho}(t) = e^{-i\omega t \text{ad}_{\hat{n}}} \left[\frac{1}{\mathcal{N}} \int \mathcal{D}^2 \xi e^{-i \int_0^t dt' \xi_i(t') R_{ij}(t') \text{ad}_{\hat{x}_j}} e^{-\frac{1}{2} \int_0^t dt' \int_0^t dt'' \xi_i(t') K_{ij}^{-1}(t', t'') \xi_j(t'')} \right] \hat{\rho}_0 \quad (10)$$

$$= e^{-i\omega t \text{ad}_{\hat{n}}} e^{-\frac{1}{2} \text{ad}_{\hat{x}_i} [\int_0^t dt' \int_0^t dt'' R_{ij}(-t') K_{jk}(t', t'') R_{kl}(t'')] \text{ad}_{\hat{x}_l}} \hat{\rho}_0. \quad (11)$$

In Eq. (11), we see the following operators showing up: $\{\text{ad}_{\hat{n}}, \text{ad}_{\hat{x}}^2 + \text{ad}_{\hat{p}}^2, \text{ad}_{\hat{x}}^2 - \text{ad}_{\hat{p}}^2, 2 \text{ad}_{\hat{x}} \text{ad}_{\hat{p}}\}$ which surprisingly, form a Lie algebra. This can be used to derive a full equation of motion for the density matrix.

Schematically, Eq. (11) can be written as

$$\hat{\rho}(t) \equiv e^{\mathcal{L}(t)} \hat{\rho}_0, \quad (12)$$

where $\mathcal{L}(t)$ is the quantum ‘Liouvillian’ in some sense. Considering our particular form of noise in Eq. (2), we can calculate

$$\text{ad}_{\hat{x}_i} \int_0^t dt' \int_0^t dt'' R_{ij}(-t') K_{jk}(t', t'') R_{kl}(t'') \text{ad}_{\hat{x}_l} = \epsilon(t) (\text{ad}_{\hat{x}}^2 + \text{ad}_{\hat{p}}^2), \quad (13)$$

where

$$\epsilon(t) \equiv \int_0^t ds \int_0^t ds' \cos \omega(s - s') k(s - s'). \quad (14)$$

$\hat{\rho} = \sum_{nm} a_{mn} |m\rangle \langle n|$ and hence we need only consider how $e^{\mathcal{L}(t)}$ acts on the state $|m\rangle \langle n|$. First, we need a some facts about operators that act in this Hilbert space. It can be shown that any operator $\hat{\mathcal{O}}$ can be expanded as

$$\hat{\mathcal{O}} = \int \frac{dy dq}{2\pi} \text{tr}[\hat{\mathcal{O}} e^{iy\hat{p} - iq\hat{x}}] e^{iq\hat{x} - iy\hat{p}}, \quad (15)$$

Next, the operators $e^{iq\hat{x} - iy\hat{p}}$ are eigenoperators of the operators $\text{ad}_{\hat{x}}$ and $\text{ad}_{\hat{p}}$:

$$\text{ad}_{\hat{x}} e^{iq\hat{x} - iy\hat{p}} = y e^{iq\hat{x} - iy\hat{p}}, \quad (16)$$

$$\text{ad}_{\hat{p}} e^{iq\hat{x} - iy\hat{p}} = q e^{iq\hat{x} - iy\hat{p}}. \quad (17)$$

And finally, we can calculate the matrix element

$$\langle n | e^{iy\hat{p} - iq\hat{x}} | m \rangle = \sqrt{\frac{n!}{m!}} (z^*)^{m-n} L_n^{(m-n)}(|z|^2) e^{-|z|^2/2}, \quad z = \frac{1}{\sqrt{2}}(y + iq), \quad (18)$$

where $L_n^{(m-n)}$ is an associated Laguerre polynomial. Also, it is easily shown that $\text{ad}_{\hat{x}}^2 + \text{ad}_{\hat{p}}^2$ commutes with $\text{ad}_{\hat{n}}$ (this can be seen from the Lie algebra). Combining these facts, we obtain from Eq. (15) that

$$e^{\mathcal{L}(t)} |m\rangle \langle n| = e^{-i\omega t(m-n)} \int \frac{dy dq}{2\pi} \langle n | e^{iy\hat{p} - iq\hat{x}} | m \rangle e^{iq\hat{x} - iy\hat{p}} e^{-\frac{1}{2}\epsilon(t)(q^2 + y^2)}. \quad (19)$$

To evaluate this, we find the matrix element

$$\langle k | \{e^{\mathcal{L}(t)} |m\rangle \langle n|\} | l \rangle = e^{-i\omega t(m-n)} \int \frac{dy dq}{2\pi} \langle n | e^{iy\hat{p} - iq\hat{x}} | m \rangle \langle k | e^{iq\hat{x} - iy\hat{p}} | l \rangle e^{-\frac{1}{2}\epsilon(t)(q^2 + y^2)}. \quad (20)$$

From Eq. (18),

$$\langle k | \{e^{\mathcal{L}(t)} |m\rangle \langle n|\} | l \rangle = \sqrt{\frac{n!l!}{k!m!}} e^{-i\omega t(m-n)} \int \frac{d^2 z}{\pi} z^{k-l} (z^*)^{m-n} L_n^{(m-n)}(|z|^2) L_l^{(k-l)}(|z|^2) e^{-(1+\epsilon(t))|z|^2}. \quad (21)$$

Now, we let $z = \sqrt{x} e^{i\theta}$ such that $d^2 z = \frac{1}{2} dx d\theta$

$$\langle k | \{e^{\mathcal{L}(t)} |m\rangle \langle n|\} | l \rangle = \sqrt{\frac{n!l!}{m!k!}} e^{-i\omega t(m-n)} \int_0^\infty dx \int_0^{2\pi} \frac{d\theta}{2\pi} x^{(k-l+m-n)/2} e^{i\theta(k-l-m+n)} L_n^{(m-n)}(x) L_l^{(k-l)}(x) e^{-(1+\epsilon(t))x} \quad (22)$$

$$= \sqrt{\frac{n!l!}{m!k!}} \delta_{k-l, m-n} e^{-i\omega t(m-n)} \int_0^\infty dx x^{m-n} L_n^{(m-n)}(x) L_l^{(m-n)}(x) e^{-(1+\epsilon(t))x}. \quad (23)$$

using the identity

$$\frac{(-x)^m}{m!} L_n^{(m-n)}(x) = \frac{(-x)^n}{n!} L_m^{(n-m)}(x), \quad (24)$$

we can write

$$\langle k | \{e^{\mathcal{L}(t)} | m \rangle \langle n | \} | l \rangle = \sqrt{\frac{m!k!}{n!l!}} \delta_{k-l, m-n} e^{-i\omega t(m-n)} \int_0^\infty dx x^{n-m} L_m^{(n-m)}(x) L_k^{(n-m)}(x) e^{-(1+\epsilon(t))x}. \quad (25)$$

The right hand side (RHS) of Eq. (25) is the same expression as the RHS of Eq. (23) with $n \leftrightarrow m$ and $k \leftrightarrow l$ (except for the multiplicative $e^{-i\omega t(m-n)}$ term). Thus, we can use Eq. (23) and assume $m \geq n$ without loss of generality. At the end of our calculation, we simply switch indices to obtain $m \leq n$.

A change of variables $y = (1 + \epsilon(t))x$ yields

$$\begin{aligned} & \langle k | \{e^{\mathcal{L}(t)} | m \rangle \langle n | \} | l \rangle \\ &= \sqrt{\frac{n!l!}{k!m!}} \frac{\delta_{k-l, m-n}}{(1 + \epsilon(t))^{m-n+1}} e^{-i\omega t(m-n)} \int_0^\infty dy y^{m-n} L_n^{(m-n)}\left(\frac{y}{1 + \epsilon(t)}\right) L_l^{(m-n)}\left(\frac{y}{1 + \epsilon(t)}\right) e^{-y}. \end{aligned} \quad (26)$$

Together with the property

$$L_n^{(m-n)}\left(\frac{y}{1 + \epsilon(t)}\right) = \frac{1}{(1 + \epsilon(t))^n} \sum_{i=0}^n \epsilon(t)^{n-i} \binom{m}{n-i} L_i^{(m-n)}(y) \quad (27)$$

we obtain

$$\begin{aligned} & \int_0^\infty dy y^{m-n} L_n^{(m-n)}\left(\frac{y}{1 + \epsilon(t)}\right) L_l^{(m-n)}\left(\frac{y}{1 + \epsilon(t)}\right) e^{-y} \\ &= \frac{1}{(1 + \epsilon(t))^{n+l}} \sum_{i=0}^n \sum_{j=0}^l \epsilon(t)^{n+l-i-j} \binom{m}{n-i} \binom{l+m-n}{l-j} \int_0^\infty dy y^{m-n} L_i^{(m-n)}(y) L_j^{(m-n)}(y) e^{-y} \end{aligned} \quad (28)$$

$$= \frac{1}{(1 + \epsilon(t))^{n+l}} \sum_{i=0}^{\min(n,l)} \epsilon(t)^{n+l-2i} \binom{m}{n-i} \binom{l+m-n}{l-i} \frac{(m-n+i)!}{i!} \quad (29)$$

$$= \frac{\epsilon(t)^{n+l}}{(1 + \epsilon(t))^{n+l}} \binom{m}{n} \binom{l+m-n}{l} (m-n)! {}_2F_1[-l, -n; 1+m-n; \epsilon(t)^{-2}], \quad (30)$$

where ${}_2F_1$ is the hypergeometric function. Thus, we can return to Eq. (26) to obtain

$$\langle k | \{e^{\mathcal{L}(t)} | m \rangle \langle n | \} | l \rangle = \sqrt{\frac{m!k!}{n!l!}} \frac{\delta_{k-l, m-n}}{(1 + \epsilon(t))^{m+l+1}} \frac{\epsilon(t)^{n+l} {}_2F_1[-l, -n; 1+m-n; \epsilon(t)^{-2}]}{(m-n)!} e^{-i\omega t(m-n)}, \quad (31)$$

expanded in the number basis, we then have

$$\begin{aligned} e^{\mathcal{L}(t)} | m \rangle \langle n | &= e^{-i\omega t(m-n)} \\ &\times \begin{cases} \sum_{l=0}^\infty \sqrt{\frac{m!(l+m-n)!}{n!l!}} \frac{\epsilon(t)^{n+l} {}_2F_1[-l, -n; 1+m-n; \epsilon(t)^{-2}]}{(m-n)!(1 + \epsilon(t))^{m+l+1}} |l+m-n\rangle \langle l| & \text{for } m \geq n, \\ \sum_{k=0}^\infty \sqrt{\frac{n!(k+n-m)!}{m!k!}} \frac{\epsilon(t)^{m+k} {}_2F_1[-k, -m; 1+n-m; \epsilon(t)^{-2}]}{(n-m)!(1 + \epsilon(t))^{n+k+1}} |k\rangle \langle k+n-m| & \text{for } n \geq m. \end{cases} \end{aligned} \quad (32)$$

ENTANGLEMENT

Consider two independent oscillators with Hamiltonian

$$H = \omega a_1^\dagger a_1 + \frac{1}{\sqrt{2}} [\xi^{(1)*}(t) a_1 + \xi^{(1)}(t) a_1^\dagger] + \omega a_2^\dagger a_2 + \frac{1}{\sqrt{2}} [\xi^{(2)*}(t) a_2 + \xi^{(2)}(t) a_2^\dagger], \quad (33)$$

where $\xi^{(n)}(t) = \xi_1^{(n)}(t) + i\xi_2^{(n)}(t)$ for $n = 1, 2$ and we impose $\langle \xi_i^{(n)}(t) \xi_j^{(m)}(t') \rangle_\xi = \delta_{nm} \delta_{ij} k(t - t')$. The initial state is $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$ and the initial density matrix is

$$\hat{\rho}(0) = \frac{1}{2}(|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|). \quad (34)$$

Now assume that $e^{\mathcal{L}_i(t)}$ propagates the density matrix for the i th oscillator forward in time by t (as given by Eq. (32)), so that $\hat{\rho}(t)$ is time-evolved by $e^{\mathcal{L}_1(t)} \otimes e^{\mathcal{L}_2(t)}$. We obtain

$$\hat{\rho}(t) = \frac{1}{2} \left(e^{\mathcal{L}_1(t)} |0\rangle\langle 0| \otimes e^{\mathcal{L}_2(t)} |1\rangle\langle 1| + e^{\mathcal{L}_1(t)} |0\rangle\langle 1| \otimes e^{\mathcal{L}_2(t)} |1\rangle\langle 0| \right. \\ \left. + e^{\mathcal{L}_1(t)} |1\rangle\langle 0| \otimes e^{\mathcal{L}_2(t)} |0\rangle\langle 1| + e^{\mathcal{L}_1(t)} |1\rangle\langle 1| \otimes e^{\mathcal{L}_2(t)} |0\rangle\langle 0| \right). \quad (35)$$

From Eq. (32) we have

$$e^{\mathcal{L}_i(t)} |0\rangle\langle 0| = \sum_{l=0}^{\infty} \frac{\epsilon(t)^l}{(1 + \epsilon(t))^{l+1}} |l\rangle\langle l|, \quad (36)$$

$$e^{\mathcal{L}_i(t)} |1\rangle\langle 1| = \sum_{l=0}^{\infty} \frac{\epsilon(t)^{l+1} (1 + l\epsilon(t)^{-2})}{(1 + \epsilon(t))^{l+2}} |l\rangle\langle l|, \quad (37)$$

$$e^{\mathcal{L}_i(t)} |0\rangle\langle 1| = e^{i\omega t} \sum_{l=0}^{\infty} \sqrt{l+1} \frac{\epsilon(t)^l}{(1 + \epsilon(t))^{l+2}} |l\rangle\langle l+1|, \quad (38)$$

$$e^{\mathcal{L}_i(t)} |1\rangle\langle 0| = e^{-i\omega t} \sum_{l=0}^{\infty} \sqrt{l+1} \frac{\epsilon(t)^l}{(1 + \epsilon(t))^{l+2}} |l+1\rangle\langle l|. \quad (39)$$

We are just interested in the entanglement between the bottom two states of each oscillator, hence if we call the projection operator $\hat{\pi} = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $\hat{\Pi} = \hat{\pi} \otimes \hat{\pi}$, explicit calculation gives the 4×4 matrix

$$\hat{\Pi} \hat{\rho}(t) \hat{\Pi} = \frac{1}{(1 + \epsilon(t))^3} \begin{pmatrix} \epsilon(t) & 0 & 0 & 0 \\ 0 & \frac{\frac{1}{2} + \epsilon(t)^2}{1 + \epsilon(t)} & \frac{1/2}{1 + \epsilon(t)} & 0 \\ 0 & \frac{1/2}{1 + \epsilon(t)} & \frac{\frac{1}{2} + \epsilon(t)^2}{1 + \epsilon(t)} & 0 \\ 0 & 0 & 0 & \frac{\epsilon(t)[1 + \epsilon(t)^2]}{(1 + \epsilon(t))^2} \end{pmatrix}. \quad (40)$$

The bath will mix all states, but Eq. (40) (properly normalized) is effectively a two-“qubit” density matrix:

$$\hat{\rho}_2(t) = \Pi \hat{\rho}(t) \Pi = \frac{\hat{\Pi} \hat{\rho}(t) \hat{\Pi}}{\text{tr } \hat{\Pi} \hat{\rho}(t) \hat{\Pi}} \quad (41)$$

$$= \frac{1}{(4\epsilon(t) + 1)(2\epsilon(t)^2 + \epsilon(t) + 1)} \quad (42)$$

$$\times \begin{pmatrix} \epsilon(t)(1 + \epsilon(t))^2 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{2} + \epsilon(t)^2\right)(1 + \epsilon(t)) & \frac{1}{2}(1 + \epsilon(t)) & 0 \\ 0 & \frac{1}{2}(1 + \epsilon(t)) & \left(\frac{1}{2} + \epsilon(t)^2\right)(1 + \epsilon(t)) & 0 \\ 0 & 0 & 0 & \epsilon(t)(1 + \epsilon(t)^2) \end{pmatrix}. \quad (43)$$

Now, we can use $\hat{\rho}_2$ to calculate the concurrence by first computing $\tilde{\rho}_2(t) = (\sigma_y \otimes \sigma_y) \hat{\rho}_2(t) (\sigma_y \otimes \sigma_y)$ and $Q(t) = \hat{\rho}_2(t) \tilde{\rho}_2(t)$. The matrix Q has eigenvalues $\lambda_i(t)$ for $i = 1, 2, 3, 4$ and where $\lambda_1(t)$ is the maximum eigenvalue. The concurrence is then

$$C(t) = \max \left\{ 0, \sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)} - \sqrt{\lambda_3(t)} - \sqrt{\lambda_4(t)} \right\}. \quad (44)$$

The result is plotted in the main paper, Fig. 1. We see sudden death of entanglement whenever $\epsilon(t) \approx 0.455090$. Note that entanglement may persist with the higher states, but our calculation suggests entanglement between $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |0\rangle$ is lost.

PROBABILITY DISTRIBUTION FUNCTIONS

In this section, we find the probability distribution functions (PDFs) of position, momentum, and energy. Consider an operator $\hat{A}(t)$, then its PDF is given by $P_{\hat{A}}[A; t] = \langle \delta[A - \langle \hat{A}(t) \rangle] \rangle_{\xi}$ where $\langle \hat{A}(t) \rangle$ is the *quantum* expectation value of \hat{A} . For example, given unspecified functions $X(t)$ and $g_i(t)$, let

$$\langle \hat{A}(t) \rangle = X(t) + \int_0^t dt' \xi_i(t') g_i(t'). \quad (45)$$

(This is the case for $\langle \hat{x}(t) \rangle$ and $\langle \hat{p}(t) \rangle$.) We then use the delta function identity $\delta(x) = \int e^{iux} (du/2\pi)$ to obtain

$$P_{\hat{A}}[A; t] = \int \frac{du}{2\pi} \frac{1}{N} \int \mathcal{D}^2 \xi(t) e^{-\frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \xi_i(t_1) K_{ij}^{-1}(t_1, t_2) \xi_j(t_2) - \int_0^t dt_1 iu \xi_i(t_1) g_i(t_1) + iu(A - X(t))}. \quad (46)$$

The above is a quadratic path integral and can thus be solved exactly by standard techniques. K^{-1} represents the inverse integral kernel of K and we have used $K_{ij}(t, t') = K_{ji}(t', t)$. Thus,

$$P_{\hat{A}}[A; t] = \int \frac{du}{2\pi} e^{-\frac{u^2}{2} \int_0^t dt_1 \int_0^t dt_2 g_i(t_1) K_{ij}(t_1, t_2) g_j(t_2) + iu(A - X(t))} \quad (47)$$

$$= \frac{1}{\sqrt{2\pi \Sigma_A(t)}} e^{-\frac{1}{2}(A - X(t))^2 / \Sigma_A(t)} \quad (48)$$

where the variance of this PDF is

$$\Sigma_A^2(t) = \int_0^t dt_1 \int_0^t dt_2 g_i(t_1) K_{ij}(t_1, t_2) g_j(t_2). \quad (49)$$

The form of g_i depends on \hat{A} (see Eq. (45)).

On the other hand, consider formally

$$\langle \hat{B}(t) \rangle = X(t) + \int_0^t dt_1 \xi_i(t_1) g_i(t_1) + \int_0^t dt_1 \int_0^t dt_2 \xi_i(t_1) F_{ij}(t_1, t_2) \xi_j(t_2). \quad (50)$$

(this is the case for energy). In this case,

$$P_{\hat{B}}[B; t] = \int \frac{du}{2\pi} \frac{1}{\sqrt{\det K}} \int \mathcal{D}^2 \xi(t) e^{-\frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \xi_i(t_1) [K_{ij}^{-1}(t_1, t_2) + 2iu F_{ij}(t_1, t_2)] \xi_j(t_2) - iu \int_0^t dt_1 \xi_i(t_1) g_i(t_1) + iu(B - X(t))} \quad (51)$$

$$= \int \frac{du}{2\pi} \sqrt{\frac{1}{\det[1 + 2iuKF]}} e^{-\frac{u^2}{2} \int_0^t dt_1 \int_0^t dt_2 g_i(t_1) [K^{-1} + 2iuF]_{ij}^{-1}(t_1, t_2) g_j(t_2) + iu(B - X(t))}. \quad (52)$$

The determinants can be viewed in the following way: To find “ $\det D(t_1, t_2)$ ” take the function $D(t_1, t_2)$ and time slice it $N - 1$ times from 0 to t , so one has an $N \times N$ matrix. Find the determinant of this matrix, then let $N \rightarrow \infty$. $1 + 2iuKF$ or K is also a 2×2 matrix, and in that case one just takes the determinant of the matrix created by the direct product of those two spaces. This whole analysis hinges on the fact that $K_{ij}(t_1, t_2) = K_{ji}(t_2, t_1)$ and further that $F_{ij}(t_1, t_2) = F_{ji}(t_2, t_1)$.

Position and momentum probability distribution function

If we let $\hat{X}_1 \equiv \hat{x}$ and $\hat{X}_2 \equiv -\hat{p}$ and we let the starting state be a coherent state $|z_0\rangle$ with $z_0 = \frac{1}{\sqrt{2}}(x_0 + ip_0)$, we have

$$\langle \hat{X}_i(t) \rangle = X_{i,\text{cl}}(t) + \int_0^t dt' R_{ij}(t - t') \xi_j(t'), \quad (53)$$

where $X_{i,\text{cl}}(t)$ are the solutions of the classical equations of motion for a harmonic oscillator, then from Eq. (48) we obtain

$$P_{\hat{x}}[X; t] = \frac{1}{\sqrt{2\pi \Sigma_x(t)}} \exp \left\{ -\frac{[X - X_{\text{cl}}(t)]^2}{2\Sigma_x(t)^2} \right\}, \quad (54)$$

$$P_{\hat{p}}[P; t] = \frac{1}{\sqrt{2\pi \Sigma_p(t)}} \exp \left\{ -\frac{[P - P_{\text{cl}}(t)]^2}{2\Sigma_p(t)^2} \right\}, \quad (55)$$

and if we use $K_{ij}(t, t') = \delta_{ij}k(t - t')$, we get $\Sigma_{x,p}^2(t) = \epsilon(t)$, where $\epsilon(t)$ is given in Eq. (14).

Number (Energy) probability distribution function

For what follows, we consider $K_{ij}(t, t') = \delta_{ij}k(t - t')$ only. Given the initial state as a number state $|n\rangle$, we have

$$\langle \hat{n}(t) \rangle = n + \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \xi_i(t_1) R_{ij}(t_1 - t_2) \xi_j(t_2), \quad (56)$$

and using Eq. (52)

$$P_{\hat{n}}[N; t] = \int \frac{du}{2\pi} \sqrt{\frac{1}{\det[1 + iuKR]}} e^{iu(N-n)}. \quad (57)$$

By functional integration methods, the determinant can be written as

$$\langle e^{-\frac{1}{2}iu\xi^T R \xi} \rangle_{\xi} = \sqrt{\frac{1}{\det[1 + iuKR]}}. \quad (58)$$

Now, let us consider the following quantity

$$\langle \langle 0 | \{ e^{-iz(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} | 0 \rangle \langle 0 | \} | 0 \rangle \rangle_{\xi} = \langle 0 | \{ \langle e^{-iz(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} \rangle_{\xi} | 0 \rangle \langle 0 | \} | 0 \rangle, \quad (59)$$

The left hand side of of Eq. (59) can be found by standard techniques

$$\langle \langle 0 | \{ e^{-iz(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} | 0 \rangle \langle 0 | \} | 0 \rangle \rangle_{\xi} = \langle e^{-\frac{1}{2}z^2 \xi^T R \xi} \rangle_{\xi}. \quad (60)$$

However, the RHS of Eq. (59) can be calculated just as in the *Density Matrix* section (see Eq. (9)) with $\Phi_i(t) \rightarrow z\Phi_i(t)$ or equivalently $\epsilon(t) \rightarrow z^2\epsilon(t)$. Reading off the $|0\rangle\langle 0|$ component in Eq. (36), we obtain

$$\langle 0 | \{ \langle e^{-iz(\Phi_1(t) \text{ad}_{\hat{x}} + \Phi_2(t) \text{ad}_{\hat{p}})} \rangle_{\xi} | 0 \rangle \langle 0 | \} | 0 \rangle = \frac{1}{1 + z^2\epsilon(t)}. \quad (61)$$

Letting $z^2 = iu$, we get

$$\sqrt{\frac{1}{\det[1 + iuKR]}} = \frac{1}{1 + iu\epsilon(t)}. \quad (62)$$

The PDF for the number operator is then

$$P_{\hat{n}}[N; t] = \int \frac{du}{2\pi} \frac{e^{iu(N-n)}}{1 + iu\epsilon(t)} = \frac{1}{i\epsilon(t)} \int \frac{du}{2\pi} \frac{e^{iu(N-n)}}{u - i/\epsilon(t)}. \quad (63)$$

Since $\epsilon(t) > 0$, this quantity is non-zero if $N > n$. Finally, we obtain

$$P_{\hat{n}}[N; t] = \frac{\Theta(N - n)}{\epsilon(t)} e^{-(N-n)/\epsilon(t)}, \quad (64)$$

as quoted in the main text.